Indian Statistical Institute, Bangalore Centre

B.Math (Hons.) III Year, Second Semester

Mid-Sem Examination - 2013-2014

Combinatorics and Graph Theory

Duration: 4 Hours February 28, 2014 Instructor: Bhaskar Bagchi

Full Marks : 100.

Note : There are five questions, each carrying 25 marks. Answer any four.

1. (a) Write down the addition and multiplication table of the field $\mathbb{F}_9$ of order 9. Hence find all the non-zero squares in this field.
   (b) Viewing $\mathbb{F}_9$ as a vector space over $\mathbb{F}_3$ of dimenson 2, construct explicity a $2-(9,3,1)$ design with point set $\mathbb{F}_9$. (Give a full list of its blocks).

   [10+15=25]

2. (a) Let D be a $2-(10,3,2)$ design. Then show that any two blocks of D meet in 1 or 2 points.
   (b) List the blocks of a $2-(10,3,2)$ design.

   [12+13=25]

3. (a) A $t-(\vartheta,k,\lambda)$ design is an incidence system on ϑ points with constant block size k such that any t distinct points are together in exactly λ blocks. Show that any $t-(\vartheta,k,\lambda)$ design is also an $s-(\vartheta,k,\lambda_s)$ design for $0 \le s \le t$ and some constant λ_s. Compute λ_s in terms of t, ϑ, k, λ.
   (b) If there is a $3-(n^2+n+2, n+2, 1)$ design then show that $n = 1, 2, 4$ or 10.

   [20+5=25]

4. (a) Define the q-factorial $n!_q$, the q binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}_q$ and prove that $\begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{n!_q}{k!_q(n-k)!_q}$ for $0 \le k \le n$.
   (b) Show that the incidence system of points and k-dimensional affine subspaces of the n-dimensional Euclidean space over $\mathbb{F}_q$ is a $2-\left(q^n, q^k, \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}_q\right)$ design for $1 \le k \le n-1$.

   [10+15=25]

5. Consider the standard action of S_n on n symbols and the induced action on the k-subsets of this n-set ($0 \le k \le n$).
   (a) Show that this induced action is transitive, compute the stabilizer of a given k-set, and hence show that the number of k-subsets is $\frac{n!}{k!(n-k)!}$
   (b) Proved that this action is imprimitive iff $n = 2k$. What are the non-trivial blocks when $n = 2k$?

   [12+12=25]